Application of Multichannel Scattering Theory to Pionic Reactions Within the Lee Model *

M. Sawicki **
Institute of Theoretical Physics, University of Warsaw

and D. Schütte
Institut für Theoretische Kernphysik der Universität Bonn

Z. Naturforsch. **36a**, 1261–1275 (1981); received June 5, 1981

The multichannel scattering theory is applied to the (2,2) sector of the (non-static) Lee model. Rigorous expressions for the transition amplitudes for two-fragments channels are derived. These expressions contain all effects of off-shell renormalization in a complete and consistent way. With suitable identification of the elementary fields of the model the reactions considered correspond to a simplified description of elastic proton-proton and pion-deuteron scattering and to pion absorption on the deuteron. We obtain a two-body equation for the description of the elastic proton-proton scattering and an extension of the two-potential formula for the pion-deuteron scattering, which can be cast into the form of the multiple-scattering series.

I. Introduction

Recently, the Lee model [1, 2] has been used for studying the structure of a field-theoretical few or many-body problem [3–5]: Schmit and Maillet [3] have investigated the NNθ sector ((2, 1) sector in the notation of Ref. [5]) as a model for π-nucleus scattering in order to test the validity of multiple scattering theory, Petry and Sawicki [4] have constructed solutions of the (2, 2) sector with the help of generalized Fadde'ev techniques whereas Ref. [5] was devoted to an analysis of renormalization corrections to the many-body ground state energy of the sector $(2n, n)$ $(n \to \infty)$.

The present paper is an extension of the investigations of Ref. [4] with the aim of constructing the S-matrix for two specific two-fragment channels of the (2, 2) sector, called V–V channel and θ–d channel (see Section 4). If the three elementary particles of the (non-static) Lee model are identified as proton, neutron and positively charged pion, the corresponding reactions represent simplified models for proton-proton scattering induced by two-pion exchange and for π^+-deuteron scattering where π^+-proton scattering is given by crossed diagrams and iterations. Also the reaction $\pi^+ + \mathrm{d} \to \mathrm{p} + \mathrm{p}$ is then described within this frame.

Application of standard multichannel scattering theory leads to formal rigorous expressions for the relevant transition operators which have to be determined from suitable extensions of the Fadde'ev equations. An analysis of the results will be used to elucidate some problems of intermediate energy physics:

i) Because the renormalization problem is rigorously solvable within the Lee model, the question of (off-shell) renormalization corrections for the scattering processes (and bound state problems) can be investigated.

ii) The role of the "new kind" of three-body potential discussed in Refs. [4, 6] (see Figure (3) can be revealed.

iii) The question of the validity of multiple scattering theory [7] discussed in Ref. [3] for the (2, 1) sector can be studied for the more complex (2, 2) sector yielding a special representation for the elastic π–d scattering amplitude (see Section 9d).

iv) The exact Lee model results for the scattering amplitudes for the reactions $\pi^+ + \mathrm{d} \to \pi^+ + \mathrm{d}$, $\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p}$ and $\pi^+ + \mathrm{d} \to \mathrm{p} + \mathrm{p}$ can be compared to those of Mizutani and Koltun [13], who treated a more realistic field theoretical model in an approximation, however, where off-shell renormalization corrections were neglected and effective coupling operators had to be introduced phenomenologically.

* Supported in part by the Deutsche Forschungsgemeinschaft.

** Institute of Theoretical Physics, University of Warsaw, Hoza 69, 00-681 Warsaw.

Reprint requests to Prof. Dr. D. Schütte, Institut für Theoretische Kernphysik der Universität Bonn, Nussallee 14 bis 16, D-5300 Bonn 1.

0340-4811 / 81 / 1200-1261 $ 01.00/0. — Please order a reprint rather than making your own copy.

We give a brief survey of our paper: in Sect. 2 we recall the definitions of the non-static Lee model and of renormalization which have been presented in detail in [5]. Section 3 presents the general formulation of the scattering theory and its application to the simplest two-body scattering cases, i.e. $\theta+\mathrm{N}\to\theta+\mathrm{N}$, $\theta+\mathrm{V}\to\theta+\mathrm{V}$, $\mathrm{N}+\mathrm{V}\to\mathrm{N}+\mathrm{V}$. In Sect. 4 we define the relevant channel states for the (2, 2)sector and derive the fundamental equation for the associated scattering state in the $\theta+\mathrm{d}$ channel using the techniques of standard multichannel scattering theory [9, 10]. In preparation of a detailed analysis of the transition operators of interest we present in section 5 a discussion of the structure of the resolvent operator in (2, 2) sector. The emerging results for the transition amplitudes for the reactions $\theta+\mathrm{d}\to\theta+\mathrm{d}$, $\mathrm{V}+\mathrm{V}\to\mathrm{V}+\mathrm{V}$ and $\theta+\mathrm{d}\to\mathrm{V}+\mathrm{V}$ are displayed in Sects. 6, 7 and 8, respectively. The structure of these results is discussed in Section 9.

II. Definition of the Lee Model and Renormalization

We use the notations of Ref. [5] which we recall very briefly: we define by $V_\alpha^+, N_\beta^+, a_k^+$ the creation operators of the fermions V, N and of the boson θ, α, β, k are compact notations for all quantum numbers to specify these particles. We use finite volume normalization and discrete sums, for convenience. The Lee model Hamiltonian is then defined by

$$H = H_0^0 + W, \tag{2.1}$$

where

$$\begin{aligned} H_0^0 &= \sum_\alpha E_\alpha^0 V_\alpha^+ V_\alpha + \sum_\beta E_\beta N_\beta^+ N_\beta \\ &\quad + \sum_k \omega_k a_k^+ a_k, \\ W &= \sum W_{\alpha\beta k}^0 V_\alpha^+ N_\beta a_k + \text{h.c.} \end{aligned} \tag{2.2}$$

The specifications of the single particle energies E_α^0, E_β, ω_k and of the interaction matrix elements $W_{\alpha\beta k}^0$ are taken from Ref. [5], the conditions on form factors contained in the quantities $W_{\alpha\beta k}^0$ are discussed in Section 9a.

Sectors (q_1, q_2) are defined by the eigenvalues q_1, q_2 of the symmetry operators Q_1, Q_2 (commuting with H) where

$$\begin{aligned} Q_1 &= \sum_\alpha V_\alpha^+ V_\alpha + \sum_\beta N_\beta^+ N_\beta, \\ Q_2 &= \sum_\alpha V_\alpha^+ V_\alpha + \sum_k a_k^+ a_k. \end{aligned} \tag{2.3}$$

The investigation of the most simple, non-trivial sector (1, 1) yields the following definition of renormalized V-particle energy E_α and renormalized interaction matrix element $W_{\alpha\beta k}$ [5]:

$$\begin{aligned} E_\alpha &= E_\alpha^0 + h_\alpha(E_\alpha), \\ h_\alpha(z) &= \sum_{\beta k} | W_{\alpha\beta k}^0 |^2/(z - E_\beta - \omega_k), \end{aligned} \tag{2.4}$$

$$\begin{aligned} W_{\alpha\beta k} &= W_{\alpha\beta k}^0 / Z_\alpha(E_\alpha), \\ Z_\alpha^2(z) &= 1 + \sum_{\beta k} | W_{\alpha\beta k}^0 |^2 / \\ &\quad \{(z - E_\beta - \omega_k) * (E_\alpha - E_\beta - \omega_k)\}. \end{aligned} \tag{2.5}$$

We assume that the parameters of the model are chosen in such a way that E_α is real. Simple off-shell renormalization corrections are described by the dressing factor $r_\alpha(z)$ given by

$$\begin{aligned} r_\alpha^{-2}(z) &= Z_\alpha^2(z)/Z_\alpha^2(E_\alpha) \\ &= 1 - (z - E_\alpha) \sum_{\beta k} | W_{\alpha\beta k} |^2 / \\ &\quad \{(E_\alpha - E_\beta - \omega_k)^2 * (z - E_\beta - \omega_k)\}. \end{aligned} \tag{2.6}$$

The bound state eigenfunction of the renormalized V-particle is

$$\begin{aligned} \psi_\alpha &= \tilde{V}_\alpha^+ |0\rangle, \quad \tilde{V}_\alpha^+ \equiv Z_\alpha^{-1}(E_\alpha) V_\alpha^+ + \Omega_\alpha, \\ \Omega_\alpha &= \sum_{\beta k} W_{\alpha\beta k}^* N_\beta^+ a_k^+ / (E_\alpha - E_\beta - \omega_k). \end{aligned} \tag{2.7}$$

It is useful to redefine the initial Hamiltonian in the following way

$$\begin{aligned} H &= H_0 + H_\mathrm{I}, \\ H_0 &= \sum_\alpha E_\alpha V_\alpha^+ V_\alpha + \sum_\beta E_\beta N_\beta^+ N_\beta \\ &\quad + \sum_k \omega_k a_k^+ a_k, \\ H_\mathrm{I} &= W - \sum_\alpha h_\alpha(E_\alpha) V_\alpha^+ V_\alpha. \end{aligned} \tag{2.8}$$

III. The "Two-Body" Sectors with one Unrenormalized Particle

1. General Formulation of the Scattering Theory

Our starting point is the application of the multichannel scattering theory [9, 10] whose conditions are fulfilled in a Lee model with massive particles: the basic concept of the theory is the definition of the "free" channel states [11]. A wave packet constructed from "free" channel states will be for large times an eigenfunction of the full Hamiltonian H describing the independent motion of two fragments since the forces between the fragments are of a particle exchange nature and, therefore, of short range. The normalization of the channel states is chosen according to the standard conditions (delta-function plus bounded, square-integrable kernel, see Eq. (III. 21) of [9]). A channel scattering state $\psi^{(\pm)}$ with correct boundary condition and normalization is then associated with a "free" channel state φ by the equation

$$|\psi^{(\pm)}\rangle = \lim_{\varepsilon\to 0} (\pm i\,\varepsilon) G(E \pm i\,\varepsilon)|\varphi\rangle , \tag{3.1}$$

where $G(z) \equiv (z-H)^{-1}$ satisfies the equations

$$\begin{aligned} G(z) &= G_0(z) + G(z) H_\mathrm{I} G_0(z) \\ &= G_0(z) + G_0(z) H_\mathrm{I} G(z) , \\ G_0(z) &= (z - H_0)^{-1} . \end{aligned} \tag{3.2}$$

For the case when (at least) one of the fragments consists of an unrenormalized particle (N or θ) one gets a simple form of the scattering state. For instance, consider a θ-particle with momentum k scattered off any target $|\psi_q\rangle$ (N, V, deuteron or nucleus) with momentum q, where $H|\psi_q\rangle = E_q|\psi_q\rangle$. The "free" channel state is then given by

$$\varphi_{kq} = a_k^+ |\psi_q\rangle . \tag{3.3}$$

We can then use the equations

$$\begin{aligned} H_0 a_k^+ &= a_k^+ (H_0 + \omega_k) , \\ G_0(z) a_k^+ &= a_k^+ G_0(z - \omega_k) , \end{aligned} \tag{3.4}$$

which by (3.2) yield

$$\begin{aligned} G(z) a_k^+ &= a_k^+ G(z - \omega_k) \\ &\quad + G(z)[H_\mathrm{I}, a_k^+] G(z-\omega_k) , \end{aligned} \tag{3.5}$$

and the basic equation for the scattering state becomes (see also [8])

$$\begin{aligned} |\psi_{k,q}^{(\pm)}\rangle &= a_k^+ |\psi_q\rangle \\ &\quad + G(E_q + \omega_k \pm i\,\varepsilon)[H_\mathrm{I}, a_k^+]|\psi_q\rangle . \end{aligned} \tag{3.6}$$

The transition operator T for $\theta - \psi$ scattering is defined by

$$\begin{aligned} &\langle \psi_{k'q'}^{(-)} | \psi_{kq}^{(+)}\rangle \\ &= \delta_{k'k}\delta_{q'q} - 2\pi i\,\delta(E_q + \omega_k - E_{q'} - \omega_{k'}) \\ &\quad \cdot \langle \varphi_{k'q'}|T(z)|\varphi_{kq}\rangle|_{z=E_q+\omega_k+i\varepsilon} , \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \langle \varphi_{k'q'}|T(z)|\varphi_{kq}\rangle &= \langle \psi_{q'}|a_k'[H_\mathrm{I}, a_k^+]|\psi_q\rangle \\ &+ \langle \psi_{q'}|[a_{k'}, H_\mathrm{I}]G(z)[H_\mathrm{I}, a_k^+]|\psi_q\rangle . \end{aligned} \tag{3.8}$$

The analogous formulation holds for $N - \psi$ scattering. Now we shall consider the most simple examples.

2. $\theta - N$ Scattering

The simplest case is $\theta - N$ scattering described within the (1, 1) sector. The "free" channel state is here

$$\varphi_{k\beta} = a_k^+ |\beta\rangle^* \tag{3.9}$$

and the transition matrix takes the form [5]

$$\begin{aligned} &\langle k'\beta'|\tau(z)|k\beta\rangle \\ &= \sum_\alpha W_{\alpha\beta'k'}^* W_{\alpha\beta k} r_\alpha^2(z)/(z - E_\alpha) , \\ &\qquad z \equiv E_\beta + \omega_k + i\,\varepsilon . \end{aligned} \tag{3.10}$$

3. $N - V$ Scattering

We consider now the (2, 1) sector, which is composed of two particle VN- and of three particle NNθ-states, with corresponding projection operators P_1 and Q_1, where

$$\begin{aligned} P_1 &= \sum_{\alpha\beta} |\alpha\beta\rangle\langle\beta\alpha| , \\ Q_1 &= 1 - P_1 = \tfrac{1}{2} \sum_{\beta\beta' k} |\beta\beta' k\rangle\langle\beta\beta' k| . \end{aligned} \tag{3.11}$$

The "free" channel state for NV scattering is

$$\varphi_{\beta\alpha} = N_\beta^+ |\psi_\alpha\rangle .$$

The commutator entering (3.6) takes now the form

$$\begin{aligned} &[H_\mathrm{I}, N_\beta^+]\,\tilde{V}_\alpha^+|0\rangle \\ &= \sum_{\alpha'\beta'} |\alpha'\beta'\rangle\langle\alpha'\beta'|B_1^{-1}(z)\,V_1(z)|\alpha\beta\rangle|_{z\equiv E_\alpha+E_\beta} , \end{aligned} \tag{3.12}$$

where the operator $B_1(z)$ is given by

$$B_1(z)|\alpha\beta\rangle = Z_\alpha^{-1}(z - E_\beta)|\alpha\beta\rangle \tag{3.13}$$

and the effective N–V potential $V_1(z)$ is defined by

$$\begin{aligned} &\langle\alpha'\beta'|V_1(z)|\alpha\beta\rangle \\ &= -\sum_k W_{\alpha'\beta k} W_{\alpha\beta' k}^* r_\alpha(z - E_\beta) r_{\alpha'}(z - E_{\beta'}) \\ &\quad \cdot \frac{1}{z - E_\beta - E_{\beta'} - \omega_k} \end{aligned} \tag{3.14}$$

(see Figure 1).

According to (3.8) and (3.12) one has now to calculate $P_1 G(z) P_1$ which after some algebraic manipulations (see Appendix A of Ref. [5]) takes the form

$$\begin{aligned} P_1 G(z) P_1 &= P_1 \cdot B_1(z) G_1(z) B_1(z) P_1 , \\ G_1(z) &\equiv (z - H_0 - V_1(z))^{-1} . \end{aligned} \tag{3.15}$$

* We use the notation $V_\alpha^+|0\rangle = |\alpha\rangle$, $N_\beta^+|0\rangle = |\beta\rangle$, $N_\beta^+ a_k^+|0\rangle = |\beta k\rangle$ etc. ($|0\rangle$ is the vacuum state) throughout this paper.

Fig. 1. Diagram yielding the effective one theta exchange potential V_1 (on-shell). The double line stands for the V-particle, a solid line for the N-particle and a dashed line for the θ-particle.

Fig. 2. Diagram yielding the effective θ-V potential V_2 (on-shell).

Consequently, the transition matrix for N–V scattering can be written as

$$\langle \varphi_{\beta'\alpha'} | T(z) | \varphi_{\beta\alpha} \rangle \equiv \langle \alpha'\beta' | t_1(z) | \alpha\beta \rangle, \quad (3.16)$$

where t_1 is the solution of the standard two-body Lippmann-Schwinger equation

$$\begin{aligned} t_1(z) &= V_1(z) + V_1(z) G_1(z) V_1(z) \\ &= V_1(z) + V_1(z) G_0(z) t_1(z). \end{aligned} \quad (3.17)$$

4. θ — V Scattering

The structure of the solutions of the scattering problem in the (1, 2) sector is quite analogous to that of the (2, 1) sector. In fact, if we would have chosen the convention that V, N and θ should be bosons of the same intrinsic type (e.g. scalars), the Lee model Hamiltonian would be invariant under the permutation $N \leftrightarrow \theta$, $E_\beta \leftrightarrow \omega_k$, $W^0_{\alpha\beta k} \leftrightarrow W^0_{\alpha k \beta}$, yielding $Q_1 \leftrightarrow Q_2$ and a mapping $(1, 2) \leftrightarrow (2, 1)$ of the two sectors under consideration. In our case, there is still a strong similarity. Consequently, channel scattering states $\psi^{\pm}_{k\alpha}$ describing θ–V scattering are associated with the channel states

$$\varphi_{k\alpha} = a_k^+ \psi_\alpha \quad (3.18)$$

through (3.6). As in the case of N–V scattering, the states $\psi^{\pm}_{k\alpha}$ define a transition operator $t_2(z)$ whose on-shell matrix elements describe θ–V scattering. The operator $t_2(z)$ can be easily shown to be a solution of the integral equation

$$t_2(z) = V_2(z) + V_2(z) G_0(z) t_2(z), \quad (3.19)$$

which is a two-body equation valid in the space of the states of the type $|\alpha k\rangle$ and where the two-body θ–V potential $V_2(z)$ is given by

$$\begin{aligned} &\langle \alpha k | V_2(z) | \alpha' k' \rangle \\ &= \sum_\beta W_{\alpha\beta k'} W^*_{\alpha'\beta k} r_\alpha(z - \omega_k) \\ &\quad \cdot r_{\alpha'}(z - \omega_{k'})/(z - E_\beta - \omega_k - \omega_{k'}). \end{aligned} \quad (3.20)$$

$V_2(z)$ corresponds to an effective θ–V potential defined through the "crossed" diagram (N-exchange graph) of Fig. 2 and includes renormalization corrections [5] via the dressing factor $r_\alpha(z)$.

IV. θ — d Scattering-Preliminary Result

The aim of our paper is to describe the reactions of the type $\theta + \mathrm{d} \to \theta + \mathrm{d}$, $\mathrm{V} + \mathrm{V} \leftrightarrow \mathrm{V} + \mathrm{V}$ and $\theta + \mathrm{d} \leftrightarrow \mathrm{V} + \mathrm{V}$, which are special cases of the reactions possible within the (2, 2) sector of the Lee model. (The d-particle will be defined below.) The Hilbert subspace corresponding to the (2, 2) sector is composed of three pieces with projection operators P, Q, R where

$$\begin{aligned} P &= \tfrac{1}{2} \sum | \alpha\alpha' \rangle \langle \alpha\alpha' |, \\ Q &= \sum | \alpha\beta k \rangle \langle \alpha\beta k |, \\ R &= \tfrac{1}{4} \sum | \beta\beta' k k' \rangle \langle \beta\beta' k k' |. \end{aligned} \quad (4.1)$$

The "free" two-fragment channel states corresponding to the scattering of a θ-particle from a d-particle with state function ψ_p are given by

$$\chi_{kp} = a_k^+ \psi_p, \quad (4.2)$$

whereas channel scattering states corresponding to the collision of two (dressed) V-particles are generated from the "free" channel states

$$\chi_{\alpha\alpha'} = \tilde{V}_\alpha^+ | \psi_{\alpha'} \rangle = \tilde{V}_\alpha^+ \tilde{V}_{\alpha'}^+ | 0 \rangle. \quad (4.3)$$

The second case will be discussed in Section 6. In order to guarantee the existence of the θ–d channel, we assume that the parameters of H are chosen in such a way that there is a bound state, called d-particle, living in the (2, 1) sector. We denote the eigenfunction and eigenvalue of this "deuteron"-like state by ψ_p and E_p, if the d-particle has total momentum p. Thus we have that the P_1-subspace projection (suitably normalized)

$$\varphi_p = B_1^{-1}(E_p) P_1 \psi_p$$

obeys the equation [12]

$$(H_0 + V_1(E_p))\varphi_p = E_p \varphi_p . \tag{4.4}$$

The eigenfunctions ψ_p are then related to φ_p through [5]

$$\psi_p = \left(1 + Q_1 \frac{1}{E_p - H_0} Q_1 W\right) P_1 B_1(E_p) \varphi_p . \tag{4.5}$$

These functions can be shown to be correctly normalized when the states φ_p are chosen to obey

$$\langle \varphi_p | 1 + n | \varphi_p \rangle = \delta_{pp'}$$

where the operator n is given by

$$\langle \alpha\beta | n | \alpha'\beta' \rangle = - \sum_k \frac{W_{\alpha\beta' k} W^*_{\alpha'\beta k} r_\alpha(E_p - E_\beta) r_{\alpha'}(E_p - E_{\beta'})}{(E_p - E_\beta - E_{\beta'} - \omega_k)^2} . \tag{4.6}$$

In order to apply (3.8) for the calculation of the θ–d scattering matrix we consider first the action of the commutator terms on ψ_p yielding

$$[H_{\mathrm{I}}, a_k^+] | \psi_p \rangle = P H_{\mathrm{I}} Q B_3 a_k^+ | \varphi_p \rangle + Q B_3^{-1} \bar{V}_{23} a_k^+ | \varphi_p \rangle \tag{4.7}$$

$\bar{V}_{23} = \bar{V}_2 + \bar{V}_3$ (we drop in most cases the argument $z \equiv E_p + \omega_k + i\varepsilon$ in the operators), where

$$\begin{aligned} B_3(z) | \alpha\beta k \rangle &= Z_\alpha^{-1}(z - E_\beta - \omega_k) | \alpha\beta k \rangle , \\ \langle \alpha'\beta' k' | \bar{V}_2(z) | \alpha\beta k \rangle &= \delta_{\beta'\beta} \langle \alpha' k' | V_2(z - E_\beta) | \alpha k \rangle , \\ \langle \alpha'\beta' k' | \bar{V}_3(z) | \alpha\beta k \rangle &= W_{\alpha\beta' k'} W^*_{\alpha'\beta k} r_{\alpha'}(z - E_{\beta'} - \omega_{k'}) r_\alpha(z - E_\beta - \omega_k) / (z - E_\beta - \omega_k - E_{\beta'} - \omega_{k'}) . \end{aligned} \tag{4.8}$$

The θ-d scattering states take the form

$$| \psi_{kp}^{(\pm)} \rangle = a_k^+ | \psi_p \rangle + G(z) \{ P H_{\mathrm{I}} Q B_3 + Q B_3^{-1} \bar{V}_{23} \} a_k^+ | \varphi_p \rangle \tag{4.9}$$

and the transition amplitude for elastic θ–d scattering appears as the matrix element of an operator $T_{\mathrm{d}}(z)$ sandwiched between pure three particle (VNθ) states $a_k^+ | \varphi_p \rangle$:

$$\begin{aligned} \langle \chi_{k'p'} | T | \chi_{kp} \rangle &= \langle a_{k'}^+ \varphi_{p'} | T_{\mathrm{d}} | a_k^+ \varphi_p \rangle \\ &= \langle a_{k'}^+ \varphi_{p'} | V_{23} + (B_3 Q H P + V_{23} B_3^{-1} Q) G(z) (Q B_3^{-1} V_{23} + P H Q B_3) | a_k^+ \varphi_p \rangle . \end{aligned} \tag{4.10}$$

The structure of the resolvent operator $G(z)$ is of general interest for our investigations and will be discussed in the next chapter.

V. The Structure of the Resolvent Operator in the (2, 2) Sector

Now we want to express all matrix elements of the full Green operator $G(z)$ in terms of renormalized quantities only. Using the projection operators P, Q, R and Eq. (3.2) we obtain a set of equations for PGP, QGP and RGP (we again drop for simplicity the energy parameter z in most places):

$$\begin{aligned} PGP &= PG_0P + PG_0PH_{\mathrm{I}}PGP + PG_0PH_{\mathrm{I}}QGP , \\ QGP &= QG_0QH_{\mathrm{I}}QGP + QG_0QH_{\mathrm{I}}PGP + QG_0QH_{\mathrm{I}}RGP , \\ RGP &= RG_0RH_{\mathrm{I}}QGP . \end{aligned}$$

After eliminating RGP one obtains

$$\begin{aligned} PGP &= PG_0P + PG_0PH_{\mathrm{I}}PGP + PG_0PH_{\mathrm{I}}QGP , \\ QGP &= QG_0Q\{H_{\mathrm{I}} + H_{\mathrm{I}}RG_0RH_{\mathrm{I}}\}QGP + QG_0QH_{\mathrm{I}}PGP . \end{aligned} \tag{5.1}$$

The operator in the bracket can be decomposed in the following way [4]:

$$QH_{\mathrm{I}}Q + QH_{\mathrm{I}}RG_0RH_{\mathrm{I}}Q = \delta h_3 + B_3^{-1}(\bar{V}_1 + \bar{V}_2 + \bar{V}_3) B_3^{-1} , \tag{5.2}$$

where all operators on the RHS of (5.2) are defined in Q-space.

$$\begin{aligned} \delta h_3(z) | \alpha\beta k \rangle &= \{ h_\alpha(z - E_\beta - \omega_k) - h_\alpha(E_\alpha) \} | \alpha\beta k \rangle \\ &= (z - H_0)(1 - B_3^{-2}(z)) | \alpha\beta k \rangle , \\ \langle \alpha'\beta' k' | \bar{V}_1(z) | \alpha\beta k \rangle &= \delta_{kk'} \langle \alpha'\beta' | V_1(z - \omega_k) | \alpha\beta \rangle , \end{aligned} \tag{5.3}$$

and the other operators are given by (4.8). We want to stress, that the operators $\bar{V}_1$ and $\bar{V}_2$ are the standard two-body interactions emerging from the corresponding VN and Vθ subsystems, whereas $\bar{V}_3$ is a new type of disconnected three body force first introduced by Stingl et al. [6] (see Figure 3).

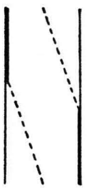

Fig. 3. Diagram yielding the effective three-body potential $\bar{V}_3$ (on-shell).

1. Modified Fadde'ev Equation in the Three-body Space

We now introduce operators $\tilde{g}_Q$, g_Q, g_3 and T_3 by

$$\tilde{g}_Q(z) = B_3 \frac{1}{z - H_0 - \bar{V}_1 - \bar{V}_2 - \bar{V}_3} B_3 \qquad (5.4)$$
$$= B_3 g_Q B_3 = B_3(g_3 + g_3 T_3 g_3) B_3 ,$$

where

$$g_3(z) = (z - H_0 - \bar{V}_3)^{-1} \qquad (5.5)$$

and T_3 satisfies the integral equation

$$T_3 = (\bar{V}_1 + \bar{V}_2) + (\bar{V}_1 + \bar{V}_2) g_3 T_3 . \qquad (5.6)$$

The resolvent $g_3(z)$ is related to the transition operator $\bar{t}_3(z)$ through

$$g_3 = G_0 + G_0 \bar{t}_3 G_0 , \qquad (5.7)$$

where $\bar{t}_3$ is the solution of

$$\bar{t}_3 = \bar{V}_3 + \bar{V}_3 G_0 \bar{t}_3 . \qquad (5.8)$$

It has been derived in Ref. [4] that the correct disconnected diagram summation of (5.8) is provided by first determining t_x by

$$t_x = \bar{V}_3 G_0 \bar{V}_3 + \bar{V}_3 G_0 \bar{V}_3 G_0 t_x , \qquad (5.9)$$

and then the operator $\bar{t}_3$ through

$$\bar{t}_3 = \bar{V}_3 + (1 + \bar{V}_3 G_0) t_x . \qquad (5.10)$$

Having (in principle) solved for $\bar{t}_3$, the suitable summation of (5.6) is performed in analogy to [4, 6]. We define

$$T_3 = T_1 + T_2 \qquad (5.11)$$

with

$$T_i = \bar{V}_i + \bar{V}_i g_3 T_3 , \quad i = 1, 2 . \qquad (5.12)$$

Since $(1 - \bar{V}_i G_0)^{-1} \bar{V}_i = \bar{t}_i$ where the operator $\bar{t}_i(z)$ ($i = 1, 2$) are given by the off-shell subsystem transition operators $t_1(z)$ and $t_2(z)$ (see Sect. 3) through

$$\langle \alpha\beta k | \bar{t}_1(z) | \alpha'\beta' k' \rangle = \langle \alpha\beta | t_1(z - \omega_k) | \alpha'\beta' \rangle \delta_{kk'} ,$$
$$\langle \alpha\beta k | \bar{t}_2(z) | \alpha'\beta' k' \rangle = \langle \alpha k | t_2(z - E_\beta) | \alpha' k' \rangle \delta_{\beta\beta} . \qquad (5.13)$$

(5.12) is equivalent to

$$T_1 = \bar{t}_1 + \bar{t}_1 G_0 T_2 + \bar{t}_1 G_0 \bar{t}_3 G_0 (T_1 + T_2) ,$$
$$T_2 = \bar{t}_2 + \bar{t}_2 G_0 T_1 + \bar{t}_2 G_0 \bar{t}_3 G_0 (T_1 + T_2) . \qquad (5.14)$$

The system of integral equations (5.14) constitutes the modified Fadde'ev equations for T_3. Since $\bar{t}_1 G_0 \bar{t}_3$ and $\bar{t}_2 G_0 \bar{t}_3$ are given by connected diagrams, these equations contain a genuine, nonsingular, three-body force as a new feature.

2. The Structure of the Resolvent $g_3(z)$

The operator $\bar{V}_3$ has a special disconnected diagram structure (see Fig. 3) which has the consequence that the resolvent g_3 can be related to the solutions of a simpler problem, namely to the resolvent of the $q_1(1) = q_2(1) = q_1(2) = q_2(2) = 1$ sector (called 1-sector) of the direct sum of two independent Lee models with the Hamiltonian

$$\bar{H} = H(1) + H(2) . \qquad (5.15)$$

Here, $H(1)$ is generated from operators $V_\alpha(1)$, $N_\beta(1)$, $a_k(1)$ ... and the matrix elements E^0_α, E_β, ω_k, $W^0_{\alpha\beta k}$ by the same prescription as in (2.1), whereas $H(2)$ is constructed correspondingly from operators $V_\alpha(2)$, $N_\beta(2)$ .... Using the abbreviation

$$B = (\beta k) , \qquad (5.16)$$

the 1-sector of the model $\bar{H}$ in (5.15) is given by the projection operators $\bar{P}$, Q_1, Q_2, $\bar{R}$ with

$$\begin{aligned}
\bar{P} &= \sum |\alpha_1\alpha_2)(\alpha_1\alpha_2| , & |\alpha_1\alpha_2) &= V^+_{\alpha_1}(1) V^+_{\alpha_2}(2) |0\rangle , \\
Q_1 &= \sum |B_1\alpha_2)(B_1\alpha_2| , & |B_1\alpha_2) &= N^+_{\beta_1}(1) a^+_{k_1}(1) V^+_{\alpha_2}(2) |0\rangle , \\
Q_2 &= \sum |\alpha_1 B_2)(\alpha_1 B_2| , & |\alpha_1 B_2) &= V^+_{\alpha_1}(1) N^+_{\beta_2}(2) a^+_{k_2}(2) |0\rangle , \\
\bar{R} &= \sum |B_1 B_2)(B_1 B_2| , & |B_1 B_2) &= N^+_{\beta_1}(1) a^+_{k_1}(1) N^+_{\beta_2}(2) a^+_{k_2}(2) |0\rangle .
\end{aligned} \qquad (5.17)$$

We have the (non-invertible) mapping I from the 1-sector of $\bar{H}$ to the (2.2) sector of H given by

$$I|\alpha_1\alpha_2) = |\alpha_1\alpha_2\rangle , \quad I|\alpha_1 B_2) = |\alpha_1 B_2\rangle = V^+_{\alpha_1} N^+_{\beta_2} a^+_{k_2} |0\rangle ,$$
$$I|B_1\alpha_2) = |B_1\alpha_2\rangle = -|\alpha_2 B_1\rangle , \quad I|B_1 B_2) = |B_1 B_2\rangle = |\beta_1\beta_2 k_1 k_2\rangle . \qquad (5.18)$$

The (unrenormalized) resolvent $\tilde{g}_3(z)$ defined by

$$\tilde{g}_3(z) = Q\frac{1}{z - H_0 - \delta h_3(z) - B_3^{-1}(z)\,\bar{V}_3(z)\,B_3^{-1}(z)}Q = QB_3(z)\,g_3(z)\,B_3(z)Q \tag{5.19}$$

is now related to the operator $\bar{g}(z)$ acting in the 1-sector, where

$$\bar{g}(z) = \bar{Q}\frac{1}{z - \bar{S}\bar{H}\bar{S}}\bar{Q}, \quad \bar{Q} = Q_1 + Q_2, \quad \bar{S} = \bar{Q} + \bar{R}, \tag{5.20}$$

by the mapping I through

$$\tilde{g}_3(z)\,I = I\bar{g}(z). \tag{5.21}$$

The proof of (5.21) is given in Appendix 1. Since

$$I\bar{W}|\alpha_1\alpha_2\rangle = W|\alpha_1\alpha_2\rangle, \quad \bar{W} = W(1) + W(2), \tag{5.22}$$

we have

$$\tilde{g}_3(z)\,W|\alpha_1\alpha_2\rangle = I\bar{g}(z)\,\bar{W}|\alpha_1\alpha_2), \tag{5.23}$$

which yields for $\alpha_1 \neq \alpha_2$

$$\langle\alpha_1\alpha_2|\,W\tilde{g}_3(z)\,W|\alpha_1\alpha_2\rangle = (\alpha_1\alpha_2|\,\bar{W}\bar{g}(z)\,\bar{W}|\alpha_1\alpha_2). \tag{5.24}$$

Here, we have made use of the fact that for $\alpha_1 \neq \alpha_2$

$$I^{-1}W|\alpha_1\alpha_2\rangle = \text{linear space spanned by } (\bar{W}|\alpha_1\alpha_2), -\bar{W}|\alpha_2\alpha_1)), \tag{5.25}$$

so that $II^{-1}W|\alpha_1\alpha_2\rangle = W|\alpha_1\alpha_2\rangle$. It has been shown in Ref. [4] that the nondiagonal matrix elements of $PW\tilde{g}_3(z)WP$ are zero so that only the matrix elements of (5.24) are relevant. Using $\bar{P}\bar{H}\bar{Q} = \bar{P}\bar{H}\bar{S}$ we see from (5.24) that

$$\langle\alpha_1\alpha_2|z - H_0^0 - PWg_3(z)\,WP|\alpha_1\alpha_2\rangle = (\alpha_1\alpha_2|z - \bar{H}_{\text{eff}}(z)|\alpha_1\alpha_2), \tag{5.26}$$

where

$$\bar{H}_{\text{eff}}(z) = \bar{P}\bar{H}\bar{P} + \bar{P}\bar{H}\bar{S}\frac{1}{z - \bar{S}\bar{H}\bar{S}}\bar{S}\bar{H}\bar{P} \tag{5.27}$$

is an effective operator in the $\bar{P}$ space. Introducing eigenfunctions Ψ_{12} of $\bar{H}$ with $\bar{H}\Psi_{12} = E_{12}\Psi_{12}$ ($E_{12} = E_{\alpha_1} + E_{\alpha_2}$) and $\bar{P}\Psi_{12} = |\alpha_1\alpha_2) = |12)$ given by (see (2.7))

$$\Psi_{12} = (1 + \Omega^0_{\alpha_1}(1))(1 + \Omega^0_{\alpha_2}(2))|12), \tag{5.28}$$
$$\Omega^0_\alpha = \sum_{\beta k} W^{0*}_{\alpha\beta k} N^+_\beta\, a^+_k\, V_\alpha/(E_\alpha - E_\beta - \omega_k),$$

we have

$$\bar{H}_{\text{eff}}(E_{12})|12) = E_{12}|12), \quad \bar{S}\Psi_{12} = (E_{12} - \bar{S}\bar{H}\bar{S})^{-1}\bar{S}\bar{H}|12). \tag{5.29}$$

Thus the lhs of (5.20) becomes zero for $z = E_{12}$ which justifies the factorization

$$P(z - H_0^0 - WQ\tilde{g}_3(z)QW)P = PA^{-1}(z)(z - H_0)A^{-1}(z)P, \tag{5.30}$$

where $A(z)$ is the diagonal P-space operator:

$$A(z)|\alpha_1\alpha_2\rangle = A_{\alpha_1\alpha_2}(z)|\alpha_1\alpha_2\rangle, \quad A^{-2}_{\alpha_1\alpha_2}(z) = 1 - (z - E_{12})^{-1} \cdot (12|\,\bar{W}(\bar{g}(z) - \bar{g}(E_{12}))\,\bar{W}|12). \tag{5.31}$$

(5.31) shows that the operator $A(z)$ has the on-shell property

$$A^{-2}_{\alpha_1\alpha_2}(E_{12}) = 1 + (12|\,\bar{W}\bar{S}(E_{12} - \bar{S}\bar{H}\bar{S})^{-2}\bar{S}\bar{W}|12) = (\Psi_{12}|\Psi_{12}) = Z^2_{\alpha_1}(E_{\alpha_1}) \cdot Z^2_{\alpha_2}(E_{\alpha_2}). \tag{5.32}$$

In order to reveal the structure of $A(z)$ in connection with renormalization, we show in the App. 2 that we have

$$\bar{g}(z)\,\bar{W}|12) = (1 - (z - E_{12})\bar{g}(z)\bar{\varphi}(z)\Omega^0_{12}|12) \tag{5.33}$$

with

$$\bar{\varphi}(z)|\alpha B) = Z^2_\alpha(z - E_\beta - \omega_k)|\alpha B),$$
$$\bar{\varphi}(z)|B\alpha) = Z^2_\alpha(z - E_\beta - \omega_k)|B\alpha),$$
$$B = \alpha k, \quad \Omega^0_{12} = \Omega^0_{\alpha_1}(1) + \Omega^0_{\alpha_2}(2).$$

This yields for the matrix elements of $A(z)$ the form

$$A^{-2}_{\alpha_1\alpha_2}(z) = 1 - (12|\,\bar{W}\bar{g}(z)\bar{\varphi}(z)\Omega^0_{12}|12) \tag{5.34}$$

and

$$A^{-2}_{\alpha_1\alpha_2}(z) - A^{-2}_{\alpha_1\alpha_2}(E_{12}) = -(z - E_{12})(12|\,\Omega^{0+}_{12}\,\bar{\varphi}(z)\bar{g}(z)\bar{\varphi}(z)\Omega^0_{12}|12). \tag{5.35}$$

(5.34) and (5.35) are also valid in the (2.2) sector of H with the substitutions

$$|12) \to |\alpha_1\alpha_2\rangle, \quad \Omega^0_{12} \to (\Omega_{\alpha_1} + \Omega_{\alpha_2}),$$
$$\bar{g}(z) \to \tilde{g}_3(z)$$

which is easily proved by making use of the operator I.

(5.35) leads to the definition of the two-body dressing factor $\bar{R}_{\alpha_1\alpha_2}(z)$ through

$$\bar{R}^{-2}_{\alpha_1\alpha_2}(z) = A^{-2}_{\alpha_1\alpha_2}(z)/A^{-2}_{\alpha_1\alpha_2}(E_{12}) = 1 - (z - E_{12})\langle\alpha_1\alpha_2|(\Omega^+_{\alpha_1} + \Omega^+_{\alpha_2}) \cdot r_3(z)g_3(z)r_3(z)(\Omega_{\alpha_1} + \Omega_{\alpha_2})|\alpha_1\alpha_2\rangle, \tag{5.36}$$

where

$$r_3(z)|\alpha\beta k\rangle = r_\alpha(z - E_\beta - \omega_k)|\alpha\beta k\rangle. \tag{5.37}$$

3. *The Effective P-space Interaction*

Now we want to make use of results of Sect. (5.2) in order to reveal the structure of the projected operator PGP.

Therefore, we first make use of the definition of $\tilde{g}_Q$ in order to get the expression for QGP given by (5.1′):

$$QGP = \tilde{g}_Q QH_{\mathrm{I}} PGP. \tag{5.38}$$

Substituting this result into the corresponding equation (5.1′) for PGP, we obtain

$$\begin{aligned} PGP &= P[z - H_0^0 - PWQ\tilde{g}_Q QWP]^{-1} P \\ &= P[z - H_0^0 - PWQB_3 g_3 B_3 QWP \\ &\quad - PWQB_3 g_3 T_3 g_3 B_3 QWP]^{-1} P. \end{aligned} \tag{5.39}$$

Using the result of (5.30) we rewrite the (5.39) in the form

$$\begin{aligned} PGP &= PA(z)(z - H_0 - V_p)^{-1} A(z) P \\ &\equiv PA(z) g_p(z) A(z) P, \end{aligned} \tag{5.40}$$

where we have introduced the effective two-body P-space operator V_p by

$$V_p = APWQB_3 g_3 T_3 g_3 B_3 QWPA. \tag{5.41}$$

The potential V_p has a completely renormalized structure and with the help of (5.33) and (5.36) can be cast into the form

$$\begin{aligned} &\langle \alpha_1 \alpha_2 | V_p(z) | \alpha_1' \alpha_2' \rangle \\ &= \langle \alpha_1 \alpha_2 | F^+(z) QT_3(z) QF(z) | \alpha_1' \alpha_2' \rangle \end{aligned} \tag{5.42}$$

with

$$\begin{aligned} F(z) | \alpha_1 \alpha_2 \rangle &= g_3(z) B_3(z) QHPA(z) | \alpha_1 \alpha_2 \rangle \\ &= R_{\alpha_1 \alpha_2}(z)((1 - (z - E_{12}) \\ &\quad \cdot g_3(z)) r_3^{-1}(z) Q \chi_{\alpha_1 \alpha_2}. \end{aligned} \tag{5.43}$$

4. *The Effective Q-space Propagator*

Now we want to find a convenient expression for the remaining projections QGQ and PGQ of resolvent $G(z)$. To this end we repeat the projection technique already presented in Chapt. V.3 and we obtain, after elimination of RGQ, the equations

$$\begin{aligned} QGQ &= QG_0 Q + QG_0 Q [H_{\mathrm{I}} + H_{\mathrm{I}} RG_0 RH_{\mathrm{I}}] \\ &\quad \cdot QGQ + QG_0 QH_{\mathrm{I}} PGQ, \\ PGQ &= PG_0 PH_{\mathrm{I}} PGQ + PG_0 PH_{\mathrm{I}} QGQ. \end{aligned} \tag{5.44}$$

Using the definitions (5.4) of $\tilde{g}_Q$ we obtain from the first equation

$$QGQ = \tilde{g}_Q + \tilde{g}_Q QH_{\mathrm{I}} PGQ, \tag{5.45}$$

which, substituted into the second equation (5.44), yields, with help of (5.40) and (5.41), the final expression for PGQ (comp. (5.38)):

$$PGQ = PA g_P APH_{\mathrm{I}} Q \tilde{g}_a. \tag{5.46}$$

This result allows us now to find a convenient representation of the operator QGQ:

$$QGQ = \tilde{g}_Q + \tilde{g}_Q QH_{\mathrm{I}} PA g_P APH_{\mathrm{I}} Q \tilde{g}_Q. \tag{5.47}$$

VI. The Final Expression for θ–d Scattering Amplitude

Now we are able to reformulate our first result, 4.10, for the θ–d transition amplitude in terms of renormalized operators obeying suitable Fadde'ev-like equations. To this end we use the results of Chapt. 5 for PGP, QGP, PGQ and QGQ and obtain the following expression for T_{d}:

$$\begin{aligned} &(a_k^+ \varphi_p = | kp \rangle), \\ &\langle kp | T_{\mathrm{d}} | k'p' \rangle = \langle kp | V_{23} + V_{23} g_Q V_{23} \\ &+ (1 + V_{23} g_Q) B_3 QHPA g_P APHQB_3 \\ &\cdot (g_Q V_{23} + 1) | k'p' \rangle. \end{aligned} \tag{6.1}$$

Analogously to the procedure of Sect. (5.1), the operator T_{d} can be constructed from the "subsystem" transition operators t_1, t_2 and from t_3 through suitable integral equations of the Fadde'ev-type. Herefore, we analyse the structure of the first part of T_{d} yielding

$$\begin{aligned} \tilde{T} &= V_{23} + V_{23} g_Q V_{23} \\ &= V_{23} + V_{23}(z - H_0 - \tilde{V}_1)^{-1} \tilde{T} \\ &= \tilde{T}_2 + \tilde{T}_3. \end{aligned} \tag{6.2}$$

Here, the operators $\tilde{T}_2$ and $\tilde{T}_3$ are given by

$$\begin{aligned} &\tilde{T}_i = \tilde{V}_i + \tilde{V}_i (z - H_0 - \tilde{V}_1)^{-1} (\tilde{T}_2 + \tilde{T}_3), \\ &i = 2, 3 \end{aligned} \tag{6.3}$$

and they are, therefore, the solutions of the following system of Fadde'ev-like equations:

$$\begin{aligned} \tilde{T}_2 &= t_2 + t_2 G_0 \tilde{T}_3 + t_2 G_0 t_1 G_0 (\tilde{T}_2 + \tilde{T}_3), \\ \tilde{T}_3 &= t_3 + t_3 G_0 \tilde{T}_2 + t_3 G_0 t_1 G_0 (\tilde{T}_2 + \tilde{T}_3). \end{aligned} \tag{6.4}$$

These equations are quite analogous to (5.14), we only have the replacement $t_1 \leftrightarrow t_3$; they also contain a nonsingular three-body force.

The transition operator T_{d} for θ–d scattering is then given in terms of $\tilde{T}$ and g_p through

$$\begin{aligned} &\langle pk | T_{\mathrm{d}} | k'p' \rangle = \langle kp | \tilde{T} + (1 + \tilde{T} g_1) \\ &\cdot B_3 QWPA g_p APWQB_3 (g_1 \tilde{T} + 1) | k'p' \rangle, \end{aligned} \tag{6.5}$$

where

$$g_1 = G_0 + G_0 t_1 G_0.$$

Note that (6.5) is formally analogous to the (2.20) of [13]. The first part $\tilde{T}$ represents all diagrams corresponding to the direct three-body scattering (involving only four-body intermediate states), whereas the second part of (6.5) describes the effects of the coupling to the VV channel. We point out, however, that our formulation includes in a systematic and consistent way all renormalization effects (i.e. both mass and coupling constant renormalization).

7. The V–V Scattering Amplitude

We consider the collision of two physical V-particles with momenta α_1 and α_2, so that the "free" channel state is given by (4.3). According to (3.1) the scattering state is generated by

$$|\psi_{\alpha_1\alpha_2}\rangle = \lim_{\varepsilon\to 0}(\pm i\varepsilon) \cdot G(E_{\alpha_1}+E_{\alpha_2}\pm i\varepsilon)\,\tilde{V}^+_{\alpha_1}|\psi_{\alpha_2}\rangle. \quad (7.1)$$

We want to commute the resolvent $G(z)$ with operator $\tilde{V}^+_\alpha$. To this end we commute first $G_0(z)$ with $\tilde{V}^+_\alpha$. Because $|\psi_\alpha\rangle$ is not an eigenstate of H_0 the commutator becomes more involved in comparison to that of (3.4). Using (2.7) for $\tilde{V}^+_\alpha$, the result is

$$G_0(z)\,\tilde{V}^+_\alpha = (\tilde{V}^+_\alpha - \Omega_\alpha) \cdot G_0(z-E_\alpha) + G_0(z)\,\Omega_\alpha. \quad (7.2)$$

Putting this into (3.2) and regrouping the terms we obtain

$$G(z)\,\tilde{V}^+_\alpha = \tilde{V}^+_\alpha \cdot G(z-E_\alpha) + G(z)\left\{[H_\mathrm{I},\tilde{V}^+_\alpha] - \sum_{\beta k} W^*_{\alpha\beta k} N^+_\beta a^+_k\right\} G(z-E_\alpha), \quad (7.3)$$

which is analogous to (3.5). Consequently, (7.1) yields for the scattering state

$$|\psi^{(\pm)}_{\alpha_1\alpha_2}\rangle = \tilde{V}^+_{\alpha_1}|\psi_{\alpha_2}\rangle + G(z)\left\{[H_\mathrm{I},\tilde{V}^+_{\alpha_1}] - \sum_{\beta k} W^*_{\alpha_1\beta_k} N^+_\beta a^+_k\right\}|\psi_{\alpha_2}\rangle\big|_{z=E_{\alpha_1}+E_{\alpha_2}\pm i\varepsilon}, \quad (7.4)$$

which generalizes (3.6).

The essential point is now the evaluation of the bracket term applied on $|\psi_{\alpha_2}\rangle$. We easily find that

$$\begin{aligned}\left\{[H_\mathrm{I},\tilde{V}^+_{\alpha_1}] - \sum_{\beta k} W^*_{\alpha_1\beta_k} N^+_\beta Q^+_k\right\}|\psi_{\alpha_2}\rangle &= H_\mathrm{I}\tilde{V}^+_{\alpha_1}|\psi_{\alpha_2}\rangle - \tilde{V}^+_{\alpha_1}H_\mathrm{I}|\psi_{\alpha_2}\rangle + \tilde{V}^+_{\alpha_2}H_\mathrm{I}|\psi_{\alpha_1}\rangle\\ &= Q(WI - I\tilde{W})\,\tilde{V}^+_{\alpha_1}(1)\,\tilde{V}^+_{\alpha_2}(2)|0\rangle, \end{aligned} \quad (7.5)$$

where we use the definitions (5.22) of $\tilde{W}$ and (5.28) of $|\psi_{12}\rangle$. The explicit calculation of the last line in (7.5) is given in Appendix 3 and yields the following result for the scattering state

$$|\psi^{(\pm)}_{\alpha_1\alpha_2}\rangle = \tilde{V}^+_{\alpha_1}|\psi_{\alpha_2}\rangle + G(z)QB_3^{-1}\,\tilde{V}_{12}\,g_3\,B_3\,WA\,|\alpha_1\alpha_2\rangle\big|_{z=E_{\alpha_1}+E_{\alpha_2}\pm i\varepsilon}. \quad (7.6)$$

Introducing the V–V S-matrix in the standard way we obtain the following result for the matrix elements of the transition operator T:

$$\langle\chi_{\alpha_1'\alpha_2'}|\,T\,|_{\alpha_1\alpha_2}\rangle = \langle\alpha_1'\alpha_2'|\,T_p\,|\alpha_1\alpha_2\rangle, \quad (7.7)$$

where

$$\langle\alpha_1'\alpha_2'|\,T_p\,|\alpha_1\alpha_2\rangle = \langle\alpha_1'\alpha_2'|\,APWQB_3g_3\,\tilde{V}_{12}\,B_3^{-1}QGQB_3^{-1}\,\tilde{V}_{12}\,g_3\,B_3QWPA\,|\alpha_1\alpha_2\rangle. \quad (7.8)$$

Taking (5.47) for QGQ and observing that the identity

$$\tilde{V}_{12}\,g_Q = T_3\,g_3 \quad (7.9)$$

holds we obtain the final form of the T-matrix elements

$$\begin{aligned}&\langle\alpha_1'\alpha_2'|\,T_p\,|\alpha_1\alpha_2\rangle\\ &= \langle\alpha_1'\alpha_2'|\,V_p + V_p g_p V_p\,|\alpha_1\alpha_2\rangle,\end{aligned} \quad (7.10)$$

where use has been made of the definitions (5.40) and (5.41). This shows that T_p is in fact the solution of the standard two-body P-space Lippmann-Schwinger equation

$$T_p = V_p + V_p G_0 T_p. \quad (7.11)$$

8. The Rearrangement Reaction $\theta + d \to V + V$

The transition matrix for the reaction $\theta + \mathrm{d} \to \mathrm{V} + \mathrm{V}$ (with initial momenta k, p and final momenta α_1, α_2 is given by

$$\langle\psi^{(-)}_{\alpha_1\alpha_2}|\psi^{(+)}_{kp}\rangle = -2\pi i\,\delta(E_{\alpha_1}+E_{\alpha_2}-E_p-\omega_k) \cdot\langle\chi_{\alpha_1\alpha_2}|\,T\,|\chi_{kp}\rangle, \quad (8.1)$$

where

$$\langle\chi_{\alpha_1\alpha_2}|\,T\,|\chi_{pk}\rangle = \langle\psi^{(-)}_{\alpha_1\alpha_2}|[H_\mathrm{I},a^+_k]|\psi_p\rangle. \quad (8.2)$$

Using (4.7) and (7.6) we may write for the right hand side of (8.2)

$$\begin{aligned}\langle\chi_{\alpha_1\alpha_2}|\,T\,|\chi_{kp}\rangle = &\langle\psi_{\alpha_2}|\,\tilde{V}_{\alpha_1}PH_{\rm I}QB_3a_k^+\,|\varphi_p\rangle + \langle\psi_{\alpha_2}|\,\tilde{V}_{\alpha_1}QB_3^{-1}\tilde{V}_{23}a_k^+\,|\varphi_p\rangle\\ &+\langle\alpha_1\alpha_2|\,PAH_{\rm I}QB_3g_3\tilde{V}_{12}B_3^{-1}QG(z)\,PH_{\rm I}QB_3a_k^+\,|\varphi_p\rangle\\ &+\langle\alpha_1\alpha_2|\,PAH_{\rm I}QB_3g_3\tilde{V}_{12}B_3^{-1}QG(z)\,QB_3^{-1}\tilde{V}_{23}a_k^+\,|\varphi_p\rangle,\end{aligned}\tag{8.3}$$

where $z\equiv E_{\alpha_1}+E_{\alpha_2}+i\,\varepsilon$.

Taking into account that

$$\begin{aligned}P\tilde{V}_{\alpha_1}^+\,|\psi_{\alpha_2}\rangle &= A\,|\alpha_1\alpha_2\rangle,\\ Q\tilde{V}_{\alpha_1}^+\,|\psi_{\alpha_2}\rangle &= B_3g_3B_3QH_{\rm I}PA\,|\alpha_1\alpha_2\rangle,\\ &(z\equiv E_{\alpha_1}+E_{\alpha_2}),\end{aligned}\tag{8.4}$$

and recalling our results for QGP and QGQ, (5.38) and (5.47), as well as (7.9) we arrive at the final form of the transition matrix:

$$\langle\psi_{\alpha_1\alpha_2}|\,T\,|\psi_{kp}\rangle\equiv\langle\alpha_1\alpha_2|\,T_r\,|kp\rangle,\tag{8.5}$$

where

$$\begin{aligned}\langle\alpha_1\alpha_2|\,T_r|\,kp\rangle = &\langle\alpha_1\alpha_2|\,(1+T_pG_0)\\ &\cdot APH_{\rm I}QB_3(1+g_1\tilde{T})\,|kp\rangle.\end{aligned}\tag{8.6}$$

Our result (8.6) is again formally equivalent to that of Mizutani and Koltun [13] (see (2.22)). The left hand part $\langle\alpha_1\alpha_2|\,(1+T_pG_0)$ is the P-space scattering wave function (i.e. the projection of the exact scattering state on the P-space with suitable normalization), whereas the term $(1+g_1\tilde{T})\,|kp\rangle$ is not a Q-space projection of the exact θ–d scattering state but contains only the contribution of the direct three-body scattering going via the four-body intermediate states (comp. (6.51)). We point out, however, that within our model the effective coupling operator $APH_{\rm I}QB_3$ obtains a well defined and renormalized form.

Discussion

a) The Limit $h_\alpha\to\infty$, $Z_\alpha^2\to\infty$, r_α Finite

Our renormalized equations make a well-defined mathematical sense for relativistic choices of the functions E_α, E_β, ω_k and $W_{\alpha\beta k}$, when $h_\alpha(z)$ and $Z_\alpha^2(z)$ become infinite. In this repect, our "Hamiltonian" formulation of the Lee-model yields very similar structures to relativistic quantum field theory — where Green's functions are used for the formulation and renormalization may be introduced just as in (2.4) and (2.5) [18]. However, it has been shown long ago [14], [16], that the transition to such a relativistic limit makes $Z_\alpha^2(E_\alpha)$ negative which is only consistent with a formulation of the Lee-model with negative metric for the V-particle — i.e. $\langle\alpha|\alpha\rangle=-1$.

This introduction of an indefinite metric still makes a mathematically well-defined theory. For our formulation of the V-scattering problems (V–θ, V–N, V–V scattering) where the V-particle takes part as asymptotic state, we have to assume that ψ_α is a bound state below N–θ threshold. This can be achieved with indefinite metric yielding a state ψ_α with positive norm. However, in contradistinction to the positive metric case, ψ_α then has necessarily a bound state companion ψ_α' with energy $E_\alpha'<E_\alpha$ and $\langle\psi_\alpha'|\psi_\alpha'\rangle<0$ which invalidates the unitarity of t_1 and t_2 even below threshold. This can be most easily verified by remarking that the renormalized equations (3.17) and (3.19) are still valid with indefinite metric, but r_α becoming imaginary in this case ($r_\alpha^{-2}(z)$ changes sign at $z=E_\alpha'$) making V_1 and V_2 nonhermitean.

A physically consistent transition to relativistic $W_{\alpha\beta k}$, yielding unitary S-matrices — is possible if $|\alpha\rangle$ is assumed to be a "bound state embedded in the continuum" of the N–θ states. In this case, E_α becomes a resonance energy when H is defined with positive definite metric, whereas with indefinite metric E_α^0 goes over into a pair of complex eigenvalues E_α, E_α^* of H with eigenfunctions of norm zero [15], [16]. However, for our formulation of scattering problems with asymptotic V-particles such "ghost states" would make no sensible theory.

Thus our formulas for all scattering problems are only meaningful if $r_\alpha(z)$ does not change sign below N–θ threshold ($r_\alpha(z)$ becomes complex above this threshold). Within the Lee-model a necessary and sufficient condition for this is that $Z_\alpha^2(E_\alpha)$ is positive which implies $Z_\alpha^2(z)$ to be finite. Thus we have to include suitable form factors in the definition of the matrix element $W_{\alpha\beta k}$ to guarantee this.

b) Some General Features of our Renormalization

The introduction of our renormalization to define V_1, V_2 and V_P is useful even in the case where h_α and Z_α are finite. We mention the following points:

i) The renormalized quantities E_α and $W_{\alpha\beta k}$ have a "natural" meaning in being directly related to physical observables.

ii) The renormalized interactions V_1, V_2 and V_P generate unitary S-matrices since they emerge directly from multichannel scattering theory.

c) Structure of the Effective Two-body Interactions

We shall discuss the cases V–θ, V–N and V–V scattering separately.

i) The effective interaction V_1 and V_2 have the important property that the on-shell quantities

$$\langle\alpha k|\, V_2(E_\alpha+\omega_k)\,|\alpha' k'\rangle \quad \text{for} \quad E_\alpha+\omega_k=E_{\alpha'}+\omega_k',$$

$$\langle\alpha\beta|\, V_1(E_\alpha+E_\beta)\,|\alpha'\beta'\rangle \quad \text{for} \quad E_\alpha+E_\beta=E_{\alpha'}+E_{\beta'}$$

are, because of $r_\alpha(E_\alpha)=1$, exactly given as contributions of "one-particle-exchange potentials" defined in the standard way [5] by the diagrams of Figs. 1 and 2 evaluated with *renormalized* energies and interactions.

The "canonical" definition of these diagrams in terms of H would involve unrenormalized quantities. Thus the usual heuristic construction of OBE potentials in terms of renormalized masses and interactions is justified from the Lee-model by our renormalization procedure.

Solving the integral equations with $V_1(z)$ or $V_2(z)$ implies the use of the off-shell vertex-renormalization function $r_\alpha(z)$ whose influence might be important [18], [6]. Such corrections could be easily incorporated even in more sophisticated models of the two-body scattering [17].

ii) The V–V potential $V_p(z)$ has quite analogous structures compared to V_1 and V_2: consider the on-shell quantity

$$\langle\alpha_1\alpha_2|\, V_p(E_{\alpha_1}+E_{\alpha_2})\,|\alpha_1'\alpha_2'\rangle \quad \text{for} \quad E_{\alpha_1}+E_{\alpha_2}=E_{\alpha_1'}+E_{\alpha_2'}$$

and take $T_3(z)$ in lowest order, i.e. $T_3=\bar{V}_1+\bar{V}_2$. One can easily verify that this potential matrix element is then given by the crossed diagrams of Fig. 4 evaluated with *renormalized* quantities. This reproduces the intuitive construction of two-boson-exchange potentials and represents a structure which would be difficult to reveal in the formulation of the effective interaction of [4]. It shows that on shell and in lowest order, the effect of V_3 — the interaction with the new type of disconnectedness [6] — is completely absorbed in taking renormalized interactions and energies for the evaluation the diagrams of Figure 4. In fact, $R_{\alpha_1\alpha_2}(z)$ and $(z-E_{12})g_3(z)r_\alpha(z)$ can be regarded as off-shell two-body vertex corrections — analogous to $r_\alpha(z)$ for V–θ and V–N scattering — whose influence could be investigated independently from solving the complete equations for $V_p(z)$. The importance of such corrections has been stressed especially by the group of Wilets [18].

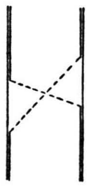

Fig. 4. Diagram yielding the lowest order on shell contribution to $V_p(z)$.

d) Structure of the θ–d Scattering

Within our formal investigation it will be most important to compare the result for the θ–d scattering matrix with that of standard multiple scattering (or Fadde'ev) theory. Within such a standard theory, the T-matrix would be given by

$$\langle pk|\,T\,|p'k'\rangle=\langle\chi_{pk}^0|\,\tau_2+\tau_3+\tau_2G_0\tau_3 + \tau_3G_0\tau_2+\tau_2G_0\tau_1G_0\tau_2 + \cdots|\,\chi_{p'k'}^0\rangle, \tag{9.1}$$

where

$$\chi_{pk}^0=a_k^+\,\varphi_p^0, \quad \varphi_0^p=\sum c_{\alpha\beta}^{op}\,|\alpha\beta\rangle.$$

φ_p^0 would be the standard d-state wave function with normalization $\langle\varphi_p^0\varphi_p^0\rangle=1$, and the subsystem transition operators $\tau_k(z)$ $(k=1,\dots,3)$ would be given by the standard two-body problems

V–N for $k=1$,
V–θ for $k=2$,
θ–N for $k=3$,

taken suitably off-shell. In our formula (6.5) the first term $\tilde{T}$ has the structure of such a series when setting $\tau_k=\bar{t}_k$, but the remainder of T_d is missing. However, with a suitable modification of the identifications, T_d can be cast into the form of the multiple scattering theory: for this, we have to put $\tau_1=\bar{t}_1$, $\tau_2=\bar{t}_2$, but the operator τ_3 has to be defined by

$$\tau_3=\bar{t}_3+(1+\bar{t}_3G_0) \cdot B_3QWPAG_0APWQB_3(1+G_0\bar{t}_3). \tag{9.2}$$

The proof of this structure of T_d is demonstrated in Appendix 4. For a further discussion we first analyse the properties of τ_3. This operator is related to the resolvent of $\bar{H}$ (see (5.15)) via

$$\tau_3 I = I \bar{\tau}_3 ,$$
$$\bar{g}_3 = g_0 + g_0 \bar{\tau}_3 g_0 , \quad g_0(z) = 1/(z - \bar{H}_0)$$
$$= \bar{Q}(z - \bar{H})^{-1} \bar{Q} . \tag{9.3}$$

Using the definition of (5.20) for $\bar{Q}$, $\bar{g}_3(z)$ can be decomposed into two terms

$$\bar{g}_3 = \bar{g}_x + \bar{g}' ,$$
$$\bar{g}_x = Q_1 \frac{1}{z - \bar{H}} Q_1 + Q_2 \frac{1}{z - \bar{H}} Q_2 , \tag{9.4}$$
$$\bar{g}' = Q_1 \frac{1}{z - \bar{H}} Q_2 + Q_2 \frac{1}{z - \bar{H}} Q_1 .$$

Using (9.3), this defines a corresponding decomposition of τ_3:

$$\tau_3 = \tau_x + \tau' .$$

τ_x and τ' are generated by the diagram-classes defined in Fig. 5a and b, respectively.

$\tau_x(z)$ is given by the N–θ T-matrix τ taken suitably (but non-canonically) off-shell:

$$\langle \alpha \beta k | \tau_x(z) | \alpha' \beta' k' \rangle$$
$$= \delta_{\alpha\alpha'} \langle \beta k | \tau(s_\alpha(z)) | \beta' k' \rangle . \tag{9.5}$$

Here, $s_\alpha(z)$ is defined through

$$\left(\alpha B \left| \frac{1}{s_\alpha(z) - H_2} \right| \alpha' B' \right) = \left(\alpha B \left| \frac{1}{z - H_1 - H_2} \right| \alpha' B' \right)$$
$$= \delta_{\alpha\alpha'} Z_\alpha^{-2}(E_\alpha) \left[\left\langle B \left| \frac{1}{z - H - E_\alpha} \right| B' \right\rangle + \sum_{B_1} \frac{|W^0_{\alpha B_1}|^2}{(E_\alpha - E_{B_1})^2} \left\langle B \left| \frac{1}{z - H - E_{B_1}} \right| B' \right\rangle \right] .$$

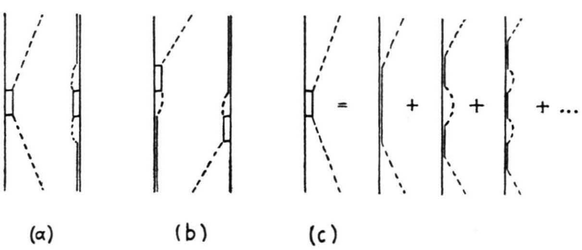

Fig. 5. Decomposition of τ_3 into τ_x and τ' in terms of two diagram classes of topologically different structure involving the off-shell θ-N transition amplitude $\tau(z)$ represented in Figure 5c.

The notations are taken from sect. 5.2, $H_k = H(k)$ $(k = 1, 2)$, $|B\rangle = |\beta k\rangle$. When $\tau(z)$ is slowly dependent on z — which is expected for realistic cases — τ_x together with τ_1 and τ_2 would generate the multiple scattering theory result for T_d in the standard form. The new feature of T_d is, therefore, represented by the additional term τ' in τ_3. The occurence of such an unconventional part has been first recognized by Stingl and Stelbovics [6]. Within the Lee-model, we obtain a well-defined, renormalized expression for $\tau'(z)$. The investigation of the influence of this term should be interesting, and this could be done within the Lee-model or — more realistically — by combining the Lee-model — $\tau'(z)$ with operators τ_1, τ_2 and τ_x representing well-adjusted fits to the corresponding subsystem scatterings.

It should be stressed, however, that we have obtained two completely equivalent representations of the θ–d scattering amplitude, (Eqs. (6.5) and (9.1)). On the one hand (9.1) is more directly connected to the standard Fadde'ev theory making use of the relevant subsystem scatterings V–N, V–θ, N–θ, where the effects of V–V scattering are generated rather indirectly by the modification $\bar{t}_3 \to \tau_3$ in (9.2) (see also Appendix 4). On the other hand the expression (6.5) for T_d exhibits a structure which is of the type of the two-potential formula [20] for the scattering matrix, where the first potential is represented by the sum $\bar{V}_1 + \bar{V}_2 + \bar{V}_3$, whereas the second potential describes the coupling to the intermediate VV states, involving the V–V scattering amplitude T_p. In this case, however, the effects of N–θ scattering are contained in higher order iterations of V_3 and in the off-shell behaviour of renormalized quantities. In addition to the unconventional 3-body term $\tau'(z)$, we have as a new feature of the θ–d scattering matrix T_d that in (4.4) φ_p is defined with normalization $\langle \varphi_p (1 + n) \varphi_p \rangle = 1$ whereas the standard theory works with $\langle \varphi_p^0 | \varphi_p^0 \rangle = 1$. Consequently, the multiple scattering result of (9.1), using φ_p^0 with $\langle \varphi_p^0 | \varphi_p^0 \rangle = 1$ has to be multiplied by $n_p n_{p'}$, where the factor n_p is defined by

$$n_p^{-2} = 1 + \sum c^{op}_{\alpha\beta} c^{*op}_{\alpha'\beta'} \langle \alpha \beta | n | \alpha' \beta' \rangle . \tag{9.6}$$

Such normalization corrections are known from standard many-body theory (see, e.g. Ref. [19]) and have to occur also for our θ–d scattering. They have to be interpreted as Pauli-principle effects from the 3-particle components of the d-state wave function and they would not appear in a standard theory of θ–d scattering.

e) Relation to the Theory of Mizutani-Koltun

When the particles V, N and θ are identified with p, n and π^+ our expressions for the scattering amplitudes for $\pi^+ + d \to \pi^+ + d$, $p + p \to p + p$, $\pi^+ + d \to p + p$ reactions are formally equivalent to the corresponding formulae of Mizutani and Koltun [13]. The nice feature of the Lee-model is that we obtain a consistent and unified prescription for calculation of all expressions appearing in the transition amplitudes. Especially, the p–p scattering is generated by the same Hamiltonian as the other reactions. Also the effects of renormalization in all propagators and effective couplings are exactly taken into account when applying the multichannel scattering theory.

Acknowledgement

One of the authors (D.S.) wishes to thank Professor L. Wilets for the hospitality extended to him during his stay at the University of Washington (Seattle).

Appendix 1

Putting $\bar{S}\bar{H}\bar{S} = \bar{Q}\bar{H}\bar{Q} + \bar{R}\bar{H}\bar{R} + \bar{Q}\bar{H}\bar{R} + \bar{R}\bar{H}\bar{Q}$ and expanding the denominator of $\bar{g}(z)$ in terms of $\bar{Q}\bar{H}\bar{R} + \bar{R}\bar{H}\bar{Q}$, we see that only even terms survive which can be resumed yielding

$$\bar{g}(z) = \bar{Q}\left(z - \bar{Q}\bar{H}\bar{Q} - \bar{Q}\bar{H}\bar{R}\,\frac{1}{z - \bar{R}\bar{H}\bar{R}}\,\bar{R}\bar{H}\bar{Q}\right)^{-1}\bar{Q}. \tag{A.1}$$

(5.21) follows from the fact that the different terms of the denominator of $\tilde{g}_3(z)$ and $\bar{g}(z)$ obey analogous commutation rules with respect to the mapping I:

$$QI = I\bar{Q}\,,\quad QH_0^0QI = I\bar{Q}(\bar{H}_0^0(1) + \bar{H}_0^0(2))\bar{Q} = I\bar{Q}\bar{H}\bar{Q}\,,$$
$$h_3(z)\,I = I(Q_1\bar{H}\bar{R}(z - \bar{R}\bar{H}\bar{R})^{-1}\bar{R}\bar{H}\bar{Q}_1 + (1 \leftrightarrow 2))\,,$$
$$B_3(z)\,\tilde{V}_3(z)\,B_3(z)\,I = I(Q_1\bar{H}\bar{R}(z - \bar{R}\bar{H}\bar{R})^{-1}\bar{R}\bar{H}\bar{Q}_2 + (1 \leftrightarrow 2))\,.$$

The proof for these relations is easily performed by taking matrix elements.

Appendix 2

In order to renormalize $\bar{g}(z)\,\bar{W}\,|\,12)$, we take out the term with $z = E_{12}$ yielding (using (5.28) and (5.29))

$$\bar{g}(z)\,\bar{W}\,|\,12) = \bar{Q}\bar{G}(z)\bar{Q}\bar{W}\,|\,12\rangle = \bar{Q}\Psi_{12} - (z - E_{12})\bar{Q}\bar{G}(z)\,\bar{S}\Psi_{12}\,, \tag{A.2}$$

where $\bar{G}(z) \equiv (z - \bar{S}\bar{H}\bar{S})^{-1}$.

Since $\bar{S} = \bar{Q} + \bar{R}$, we obtain by expanding and resuming

$$\bar{Q}\bar{G}(z)\,\bar{S} = \bar{Q}\bar{G}(z)\bar{Q} + \bar{Q}\bar{G}(z)\bar{Q}\bar{H}\bar{R}\bar{G}(z)\,\bar{R}\,.$$

Applied to $\Psi_{12} = (1 + \Omega^0_{\alpha_1}(1))\,(1 + \Omega^0_{\alpha_2}(2))\,|12)$, we get with $\Omega^0_{\alpha_i}(i) = \Omega_i \quad (i = 1, 2)$

$$\begin{aligned}\bar{Q}\bar{G}(z)\,\bar{S}\Psi_{12} &= \bar{g}(z)\,(\Omega_1 + Q_1HR(z - RHR)^{-1}\,R\Omega_1\Omega_2 + (1 \leftrightarrow 2))\,|\,12)\\ &= \bar{g}(z)\sum_{B_1} W^0_{\alpha_1 B_1}/(E_{\alpha_1} - E_{B_1})\left(1 + \sum_{B_2}|\,W_{\alpha_2 B_2}|^2/(E_{\alpha_2} - E_{B_2})/(z - E_{B_1} - E_{B_2})\right)|\,B_1\alpha_2)\\ &\quad + (1 \leftrightarrow 2) = \bar{g}(z)\,\bar{\varphi}(z)\,(\Omega_1 + \Omega_2)\,|\,12)\,.\end{aligned}$$

Here, we have used the convention $B = \beta k$, $E_B = E_\beta + \omega_k$, $W^0_{\alpha B} = W^0_{\alpha\beta k}$. Inserting this result into (A.2) yields (5.33).

Appendix 3

In order to evaluate the expression (7.5) we observe first, that

$$Q(WI - I\bar{W})\,\tilde{V}^+_{\alpha_1}(1)\,\tilde{V}^+_{\alpha_2}(2)\,|\,0\rangle \equiv Q(WRI - I\bar{W}\bar{R})\,\tilde{V}^+_{\alpha_1}(1)\,\tilde{V}^+_{\alpha_2}(2)\,|\,0\rangle\,.$$

Now, by (5.29) we have

$$\bar V^+_{\alpha_1}(1)\,\bar V^+_{\alpha_2}(2)\,|0\rangle \equiv (\bar P+\bar S)\,\bar V^+_{\alpha_1}(1)\,\bar V^+_{\alpha_2}(2)\,|0\rangle = \left(1+\bar S\frac{1}{E_{12}-\bar S\bar H\bar S}\bar S\bar W\right)\bar P A(E_{12})\,|\alpha_1\alpha_2\rangle$$
$$= (1+\bar S\bar G(E_{12})\bar S\bar W)\,\bar P A(E_{12})\,|\alpha_1\alpha_2\rangle\,,$$

where $\bar G(z)=1/(z-\bar S\bar H\bar S)$. The first equation takes now the form

$$Q(WI-I\bar W)\,\bar V^+_{\alpha_1}(1)\,\bar V^+_{\alpha_2}(2)\,|0\rangle = Q(WI-I\bar W)\,\bar R\bar G(E_{12})\bar Q\bar W\bar P A(E_{12})\,|\alpha_1\alpha_2)\,.$$

We introduce now a free resolvent $\bar G_0(z)$, $\bar G_0(z)=1/(z-\bar S\bar H_0\bar S)$, and observe that

$$\bar R\bar G(z)\bar Q = \bar R\bar G_0(z)\,\bar R\bar W\bar Q\bar G(z)\bar Q = \bar R\bar G_0(z)\,\bar R\bar W\bar g(z)\,,$$

which yields

$$Q(WI-I\bar W)\,\bar V^+_{\alpha_1}(1)\,\bar V^+_{\alpha_2}(2)\,|0\rangle = Q(WI-I\bar W)\,\bar R\bar G_0(E_{12})\,\bar R\bar W\bar Q\bar g(E_{12})\,\bar W\bar P\bar A\,|\alpha_1\alpha_2)\,.$$

Since

$$Q I\bar W\bar R\bar G_0\,\bar R\bar W\bar Q = Q(h_3+B_3^{-1}\,\bar V_3\,B_3^{-1})\,I\bar Q\,,$$
$$Q W I\bar R\bar G_0\,\bar R\bar W Q = Q(h_3+B_3^{-1}(\bar V_3+\bar V_1+\bar V_2)\,B_3^{-1})\,I\bar Q\,,$$

we obtain with help of (5.23) the result

$$Q(WI-I\bar W)\,\bar V^+_{\alpha_1}(1)\,\bar V^+_{\alpha_2}(2)\,|0\rangle = QB_3^{-1}\,\bar V_{12}\,g_3\,B_3\,WA\,|\alpha_1\alpha_2\rangle\,,$$

which completes our proof.

Appendix 4

We demonstrate now the equivalence of (9.1) and (6.5). We introduce the operator T given by solution of the standard Fadde'ev equations

$$T = T_1+T_2+T_3\,,\quad T_i = t_i + t_i G_0 \sum_{k\neq i} T_{k'}\,,\quad i,k=1,2,3\,, \tag{A.4}$$

where t_i are given by (5.13) and (5.10) and note that the following relation holds:

$$T = (1+t_1G_0)\,\tilde T\,(1+G_0t_1)+t_1 = (1+t_3G_0)\,T_3\,(1+G_0t_3)+t_3\,. \tag{A.5}$$

Here, $\tilde T$ and T_3 are given by (6.2) and (5.11), respectively. We therefore observe that the operator $\tilde T_{\rm d}$ given by

$$\tilde T_{\rm d} = (1+t_1G_0)\,T_{\rm d}\,(1+G_0t_1)+t_1 \tag{A.6}$$

can be represented by the expansion

$$\tilde T_{\rm d} = T+(1+TG_0)\,\{U+U[G_0(T-t_3)G_0U]+U[G_0(T-t_3)G_0U]$$
$$\cdot[G_0(T-t_3)G_0U]+\cdots\}\,(1+G_0T)\,, \tag{A.7}$$

where $T_{\rm d}$ is given by (6.5) and

$$U = B_3QWPAG_0APWQB_3\,. \tag{A.8}$$

The validity of (A.7) can be verified by looking at the structure of the expansion for the g_p

$$g_p = G_0+G_0V_pG_0+G_0V_pG_0V_pG_0+\cdots\,, \tag{A.9}$$

where V_p, defined by (5.41), can also be written as

$$V_p = APWQB_3G_0(T-t_3)G_0B_3QWPA\,. \tag{A.10}$$

Our aim is to show that the transition amplitude $\tilde{T}_d$ can also be generated from the standard Fadde'ev equations, i.e. we have

$$\tilde{T}_d = \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3, \tag{A.11}$$

where

$$\tilde{T}_i = \tau_i + \tau_i G_0 \sum_{i \neq k} \tilde{T}_k, \quad i, k = 1, 2, 3, \tag{A.12}$$

and

$$\tau_1 = \bar{t}_1, \quad \tau_2 = \bar{t}_2, \quad \tau_3 = t_3 + (1 + t_3 G_0) U (1 + G_0 t_3) \equiv t_3 + t_3'. \tag{A.13}$$

For the proof of (A.11) we replace U by λU and we expand the solution of (A.12) in powers of λ:

$$\tilde{T}_k = \sum_\mu \lambda^\mu T_k^{(\mu)}. \tag{A.14}$$

In the lowest order we obtain for $T_k^{(0)}$ just (A.4), so that

$$T^{(0)} \equiv T_1^{(0)} + T_2^{(0)} + T_3^{(0)} = T. \tag{A.15}$$

The evaluation of higher orders of the expansion yields

$$T_1^{(\mu)} = t_1 G_0 (T_2^{(\mu)} + T_3^{(\mu)}), \quad T_2^{(\mu)} = t_2 G_0 (T_1^{(\mu)} + T_3^{(\mu)}),$$
$$T_3^{(\mu)} = t_3 G_0 (T_1^{(\mu)} + T_2^{(\mu)}) + t_3' \cdot \delta_{\mu,1} + t_3' G_0 (T_1^{(\mu-1)} + T_2^{(\mu-1)}). \tag{A.16}$$

We multiply the rows of (A.16) by $(1 - V_1 G_0)$, $(1 - V_2 G_0)$ and $(1 - V_3 G_0)$, respectively, and add them together yielding

$$T^{(\mu)} \equiv T_1^{(\mu)} + T_2^{(\mu)} + T_3^{(\mu)} = V G_0 T^{(\mu)} + (1 - V_3 G_0)(t_3' \cdot \delta_{\mu 1} + t_3' G_0 (T_1^{(\mu-1)} + T_2^{(\mu-1)})), \tag{A.17}$$

where $V = \bar{V}_1 + \bar{V}_2 + \bar{V}_3$. Since $(1 - V G_0)^{-1} = (1 + T G_0)$, (A.17) yields the recurrence relation

$$T^{(\mu)} = (1 + T G_0) U (1 + G_0 t_3) [G_0 (T_1^{(\mu-1)} + T_2^{(\mu-1)}) + \delta_{\mu 1}]. \tag{A.18}$$

(A.18) has the solution

$$T^{(\mu)} = (1 + T G_0) U (G_0 (T - t_3) G_0 U)^{(\mu-1)} (1 + G_0 T), \tag{A.19}$$

which is immediately verified by induction. Consequently, the sum $\sum_\mu T^{(\mu)}$ is equal to $\tilde{T}_d$ given by (A.7) which completes our proof. Since the θ–d scattering amplitude T_d is related to the total amplitude $\tilde{T}_d$ by (A.5) one obtains for T_d the multiple scattering series (9.1).

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[2] S. S. Schweber, An Introduction into Relativistic Quantum Field Theory, Chapter 12, Peterson & Co., Evanstone 1961.
[3] C. Schmit and J.-P. Maillet, Nucl. Phys. **A312**, 236 (1978).
[4] H. R. Petry and M. Sawicki, Z. Naturforsch. **33a**, 1 (1978).
[5] D. Schütte and J. da Providencia, Nucl. Phys. **A338**, 463 (1980).
[6] M. Stingl and A. T. Stelbovics, J. Physics G, **43**, 1371, 1389 (1978).
[7] M. L. Goldberger and K. M. Watson, Collision Theory, Wiley, New York 1964, C.
[8] G. Miller, Phys. Rev. **C16**, 2325 (1977).
[9] H. Eckstein, Phys. Rev. **101**, 880 (1956).
[10] C. G. Wick, Rev. Mod. Phys. **27**, 339 (1955).
[11] W. Sandhas, General Concepts of Multichannel Collision Theory, Trieste Lecture Notes 1978.
[12] H. Feshbach, Ann. Phys. New York **5**, 357 (1958); **19**, 287 (1962).
[13] T. Mizutani and D. S. Koltun, Ann. Phys. **109**, 1 (1977).
[14] W. Pauli and G. Källen, Kgl. Danske. Selk., Mat. Fys. Medd. **30**, 3 (1955).
[15] W. Heisenberg, Introduction into the Field Theory of Elementary Particles, Intersciene 1966.
[16] T. D. Lee and G. C. Wick, Nucl. Phys. B **9**, 209 (1969).
[17] K. Kotthoff et al., Nucl. Phys. **A247**, 429 (1975).
[18] L. Wilets, Green's Function Method for the Relativistic Field Theory Man-body Problem, in "Mesons in Nuclei", Vol. II, North-Holland, Amsterdam 1979.
[19] T. Fliessbach, Z. Phys. **A288**, 211 (1978).
[20] Ref. [7], Chapter 5.4.